The Lossy Waveguide as a Problem in Perturbation Theory

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Abstract —**A systematic perturbation theory for lossy waveguides is presented. Two possible small dimensionless parameters are identified and introduced into Maxwell's equations by rescaling pursuant to effecting an expansion of the fields in these parameters.**

I. Introduction

Very few waveguide problems can be solved exactly if the guide walls are taken to have finite conductivity. As a consequence, some approximation procedure is needed to obtain the dispersion relation for this type of guide. Most treatments do not, however, present a systematic scheme for effecting such an approximation [1]–[4]. (In [1] elements of such a scheme are suggested.) Several are heuristic while others involve formal expansions in a parameter that is then set equal to unity. In addition, no attempt is made to identify small dimensionless parameters associated with the guide so that in general one cannot easily assess the effectiveness of these treatments.

Perturbation theory for a lossy waveguide is more complicated than standard perturbation theory. The latter treatment involves a straightforward expansion in some small parameter that appears explicitly in the equation or equations one wishes to solve. In the case of a waveguide there are no immediately evident small parameters appearing in the Maxwell equations that one uses to determine the fields in the guide and its walls. They arise only after one has identified the small dimensionless parameters that are associated with the guide in question and one has rescaled the coordinates appearing in these equations. As we will see, both these steps require some thought concerning the properties of the guide.

II. Physical Considerations

The physical considerations that lead to the construction of small dimensionless parameters involve the cross-sectional dimensions of the guide, which we characterize by L, the smallest of these dimensions, the skin depth δ and conductivity σ in the walls, and the angular frequency ω of the wave in the guide. The skin depth is given by

$$\delta=\sqrt{\frac{2}{\mu_c\omega\sigma}} \tag{1}$$

where μ_c is the magnetic permeability of the wall. (We use SI units throughout.) In a typical guide, L will be much larger than δ; hence one small parameter we can form is $\epsilon_1=\delta/L$. If indeed ϵ_1 is small compared to unity, then we can expect that changes in the field in a wall in a direction normal to its surface will be large compared to changes parallel to this surface.

A possible second small parameter involves the ratio of ω to σ. This is the ratio of the charge relaxation time in the conductor to the period of oscillation in the guide and is also small in a typical guide. In what follows we will take our second small parameter to be $\epsilon_2=\sqrt{\epsilon_c\omega/\sigma}$, where ϵ_c is the dielectric constant in the wall. Note that both ϵ_1 and ϵ_2 are dimensionless; hence it makes sense to say that they are small compared to unity. While in principle neither parameter need be small, in practice they always are.

III. Maxwell's Equations and Boundary Conditions

Three of Maxwell's equations have the same form in the guide and in its walls. They are

$$\nabla\cdot\boldsymbol{D}=0 \tag{2a}$$
$$\nabla\cdot\boldsymbol{D}^c=0 \tag{2b}$$
$$\nabla\cdot\boldsymbol{B}=0 \tag{3a}$$
$$\nabla\cdot\boldsymbol{B}^c=0 \tag{3b}$$

and

$$\nabla\times\boldsymbol{E}+\partial_t\boldsymbol{B}=0 \tag{4a}$$
$$\nabla\times\boldsymbol{E}^c+\partial_t\boldsymbol{B}^c=0 \tag{4b}$$

where the superscript c refers to the conductor and where ∂_t is short for $\partial/\partial t$. The fourth Maxwell equation in the guide has the form

$$\nabla\times\boldsymbol{H}-\partial_t\boldsymbol{D}=0 \tag{5a}$$

while in the walls it has the form

$$\nabla\times\boldsymbol{H}^c-\partial_t\boldsymbol{D}^c=\sigma\boldsymbol{E}^c \tag{5b}$$

where we have used Ohm's law in the form $\boldsymbol{J}=\sigma\boldsymbol{E}$.

As they stand, these equations do not involve either ϵ_1 or ϵ_2 explicitly. However, if we assume a time dependence for the fields of the form $e^{j\omega t}$, (5b) can be rewritten as

$$\nabla\times\boldsymbol{H}^c=\sigma\left(1+j\epsilon_2^2\right)\boldsymbol{E}^c \tag{6}$$

which does exhibit an explicit dependence on ϵ_2.

Exhibiting a dependence on ϵ_1 requires more work and involves recognizing that spatial changes in the fields in the conductor in a direction normal to its surface occur on a scale δ while those in directions parallel to this surface change on a scale L. To make explicit this observation we decompose ∇ into a part that is normal and a part that is parallel to the conductor's surface according to

$$\nabla=-\boldsymbol{n}\partial_\xi+\nabla_\parallel \tag{7}$$

where $\boldsymbol{n}$ is the normal to the wall surface and points into the guide while ξ increases into the wall and where $\boldsymbol{n}\cdot\nabla_\parallel=0$. (Since by assumption, $\epsilon_1\ll 1$ we can neglect curvature effects in our treatment.) In a like manner we can decompose the fields in the wall into normal and parallel components according to

$$\boldsymbol{E}^c=\boldsymbol{n}E_\perp+\boldsymbol{E}_\parallel^c \tag{8}$$

and similarly for $\boldsymbol{B}^c$. Using these decompositions and keeping in mind the assumed time dependence of the fields, (4b) and (6) can be put into the forms

$$-\boldsymbol{n}\times\partial_\xi\boldsymbol{E}_\parallel^c+\nabla_\parallel\times\left(E_\perp^c\boldsymbol{n}+\boldsymbol{E}_\parallel^c\right)=-j\omega\left(B_\perp^c\boldsymbol{n}+\boldsymbol{B}_\parallel^c\right) \tag{9}$$

and

$$-\boldsymbol{n}\times\partial_\xi\boldsymbol{H}_\parallel^c+\nabla_\parallel\times\left(H_\perp^c\boldsymbol{n}+\boldsymbol{H}_\parallel^c\right)=\sigma\left(1+j\epsilon_2^2\right)\left(E_\perp^c\boldsymbol{n}+\boldsymbol{E}_\parallel^c\right). \tag{10}$$

We now make the assumption that the fields in the walls depends on ξ and on η, the coordinates parallel to the wall surface, through the combinations ξ/δ and η/L. This assumption formalizes the statement that changes normal to the wall are large compared to those parallel to it if $\delta/L\ll 1$. Thus the fields can be considered to be functions of new coordinates $\hat{\xi}=\xi/\delta$ and $\hat{\eta}=\eta/L$. Derivatives with respect to these "hatted" variables will therefore not change the order of magnitude of a

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function as far as its dependence on ϵ_1 and ϵ_2 is concerned. In terms of these "stretched" coordinates, (9) and (10) take the forms

$$-\boldsymbol{n}\times\partial_\xi \boldsymbol{E}_{\|}^c+\epsilon_1\hat{\nabla}_{\|}\times\left(E_\perp^c\boldsymbol{n}+\boldsymbol{E}_{\|}^c\right)=-j\sqrt{2\mu_c/\epsilon_c}\,\epsilon_2\left(H_\perp^c\boldsymbol{n}+\boldsymbol{H}_{\|}^c\right) \tag{11}$$

and

$$-\boldsymbol{n}\times\partial_\xi \boldsymbol{H}_{\|}^c+\epsilon_1\hat{\nabla}_{\|}\times\left(H_\perp^c\boldsymbol{n}+\boldsymbol{H}_{\|}^c\right)=\sqrt{2\epsilon_c/\mu_c}\,\frac{1}{\epsilon_2}\left(1+j\epsilon_2^2\right)\left(E_\perp^c\boldsymbol{n}+\boldsymbol{E}_{\|}^c\right). \tag{12}$$

From (12) it follows by equating parallel components that, in lowest approximation,

$$-\boldsymbol{n}\times\partial_\xi \boldsymbol{H}_{\|}^c=\sqrt{2\epsilon_c/\mu_c}\,\frac{1}{\epsilon_2}\boldsymbol{E}_{\|}^c. \tag{13}$$

We see from this equation that if $\boldsymbol{H}_{\|}^c=O(1)$, then $\boldsymbol{E}_{\|}^c=O(\epsilon_2)$, where O denotes the standard order symbol. It also follows from (12) by equating normal components that $E_\perp=O(\epsilon_1\epsilon_2)$. By equating parallel components of (11) it next follows that

$$-\boldsymbol{n}\times\partial_\xi \boldsymbol{E}_{\|}^c=-j\sqrt{2\mu_c/\epsilon_c}\,\epsilon_2\boldsymbol{H}_{\|}^c. \tag{14}$$

And finally it follows again from (11) that $H_\perp^c=O(\epsilon_1/\epsilon_2)E_{\|}^c=O(\epsilon_1)$. These results allow us to determine the starting point in a double expansion in ϵ_1 and ϵ_2 of the fields in the walls of the guide. The continuity in $\boldsymbol{H}_{\|}$, $\boldsymbol{E}_{\|}$, and $B_\perp$ at the wall boundaries then allows us to determine the starting points for the expansions of these fields in the guide.

In order to determine these boundary fields we combine (13) and (14) to obtain

$$\partial_\xi^2\boldsymbol{H}_{\|}^c=-2j\boldsymbol{H}_{\|}^c. \tag{15}$$

This equation has as its solution

$$\boldsymbol{H}_{\|}^c=\boldsymbol{H}_{0\|}^c e^{-(1+j)\xi} \tag{16}$$

where $\boldsymbol{H}_{0\|}$ is the value of $\boldsymbol{H}_{\|}^c$ at the boundary of the conductor. Because of the continuity in $\boldsymbol{H}_{\|}$ it follows that $\boldsymbol{H}_{0\|}=\boldsymbol{H}_{0\|}^c$. Hence $\boldsymbol{H}_{\|}^c$ can be determined from a knowledge of $\boldsymbol{H}_{\|}$ in the guide. From (14) it next follows that

$$\boldsymbol{E}_{\|}^c=(1+j)\sqrt{\mu_c/2\epsilon_c}\,\epsilon_2\left(\boldsymbol{n}\times\boldsymbol{H}_{\|}^c\right). \tag{17}$$

We can now use this result and the continuity in $\boldsymbol{E}_{\|}$ to determine the boundary condition on this quantity in the guide.

IV. Guide Fields and Dispersion Relations

In a guide with walls of infinite conductivity the dispersion relation between wavenumber k and frequency ω is gotten by solving a two-dimensional Helmholtz equation. Boundary conditions on the walls of the guide appropriate for TM and TE modes lead to an eigenvalue problem which gives the desired dispersion relation. These boundary conditions are obtained from the continuity conditions at the walls plus the requirement that all fields vanish in the conductor. In the case of finite conductivity this is no longer the case and we must use the results of the previous section to obtain the appropriate conditions. To see how these latter conditions lead to a dispersion relation we will consider the case of TM modes in a cylindrical guide with the axis in the z direction.

The basic field in such a guide is $\psi\equiv E_z$, and it satisfies the equation

$$\left(\nabla_T^2+\gamma^2\right)\psi=0 \tag{18}$$

where $\nabla_T^2=\nabla^2-\partial_z^2$ and $\gamma^2\equiv\mu\epsilon\omega^2-k^2$ and where we have assumed the z dependence for the fields to be e^{-jkz}. In the infinite conductivity case the boundary condition is $\psi=0$ on the walls. From (17) and the continuity in E_z, it follows that ψ_0 is no longer zero in the finite conductivity case but rather is given by

$$\psi_0=E_{0z}=(1+j)\sqrt{\mu_c/2\epsilon_c}\,\epsilon_2\left(\boldsymbol{n}\times\boldsymbol{H}_{0\|}\right)_z. \tag{19}$$

One can now use Maxwell's equations in the guide to relate $\boldsymbol{n}\times\boldsymbol{H}_{\|}$ to $\nabla_T\psi$ (see e.g. [1] for details) to obtain the result

$$\psi_0=\frac{1}{2}\epsilon_1\frac{\mu_c}{\mu}(1-j)\left(1+\frac{k^2}{\gamma^2}\right)\partial_\xi\psi_0. \tag{20}$$

The form of this boundary condition suggests that, at least in the lowest level of approximation, we take

$$\psi=\psi^{(0)}+\epsilon_1\psi^{(1)}+\cdots \tag{21}$$

and

$$\gamma=\gamma^{(0)}+\epsilon_1\gamma^{(1)}+\cdots. \tag{22}$$

When these expansions are substituted into (18) and (20) and the coefficients of powers of ϵ_1 equated, we obtain the equations

$$\left(\nabla_T^2+\gamma^{(0)2}\right)\psi^{(0)}=0\qquad\psi_0^{(0)}=0 \tag{23}$$

and

$$\left(\nabla_T^2+\gamma^{(0)2}\right)\psi^{(1)}+2\gamma^{(0)}\gamma^{(1)}\psi^{(0)}=0$$

$$\psi_0^{(1)}=\frac{1}{2}\frac{\mu_c}{\mu}\left(\frac{\omega}{\omega_0}\right)^2(1-j)\,\partial_\xi\psi_0^{(0)} \tag{24}$$

where ω_0 is the cutoff frequency of the unperturbed mode. We see that equations (23) are just those for the infinite conductivity case; hence we can conclude that the infinite conductivity result is indeed a good first approximation to the exact result.

In order to solve equations (24) for $\psi^{(1)}$ we need to know $\gamma^{(1)}$. We can calculate this quantity by making use of Green's theorem in two dimensions:

$$\int_A\left[\phi\nabla_T^2\psi-\psi\nabla_T^2\psi\right]da=\oint_C\left[\psi\partial_n\phi-\phi\partial_n\psi\right]dl \tag{25}$$

where the line integral is around the curve C bounding the area A, which, in our case, we take to be the cross section of the guide. Let us now take, in this equation, $\phi=\psi^{(0)*}$, where $*$ denotes complex conjugate. It then follows from (18) and (23) and the fact that $\psi^{(0)}$ vanishes on C that

$$\left(\psi^{(0)2}-\gamma^2\right)\int_A\psi^{(0)*}\psi\,da=\oint_C\psi\,\partial_n\psi^{(0)*}\,dl. \tag{26}$$

If we now substitute our expansions (21) and (22) into this equation we obtain the final result that

$$-2\gamma^{(0)}\gamma^{(1)}=\frac{1}{2}\left(\frac{\omega}{\omega_0}\right)^2(1-j)\frac{\oint_C\left|\partial_n\psi^{(0)}\right|^2dl}{\int_A\left|\psi^{(0)}\right|^2da} \tag{27}$$

which determines $\gamma^{(1)}$ in terms of quantities found in the zeroth approximation and agrees with the result found in [1].

For most practical purposes it is sufficient to determine γ to this order of accuracy in order to calculate the attenuation in the guide. If necessary, however, we can substitute this result back into (24) in order to determine $\psi^{(1)}$ and so calculate the fields in the guide to $O(\epsilon_1)$. These fields then supply boundary conditions via the continuity equations for the calculation of the next approximation to the fields in the conductor, which in turn can be used as boundary conditions for the determination of the next approximation to the fields in the guide, and so on. In carrying out this procedure the proper ordering of terms will be automatic with the field equations in the form given here.

V. Summary

A consistent approximation scheme for lossy waveguides has been developed which can, in a straightforward manner, be extended to any order of accuracy desired. Although two small parameters are involved in the expansion of the fields in the guide and its walls, the field equations are shown to force the correct ordering of parameters in this expansion.

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Expansions for the Capacitance of a Cross Concentric with a Circle with an Application

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Abstract **—Expansions are given for the capacitance per unit length for the geometry having a cross section in which an equiarmed cross is concentric with an external circle.**

I. Introduction

Oberhettinger and Magnus [1, p. 61] have considered the problem of determining the capacitance of a coaxial structure whose outer conductor has a circular cross section while the cross section of the inner conductor is a line through the axis. The problem considered here differs in that the inner conductor has a cross section which is a symmetric cross centered on the axis of the outer circular conductor as shown in the z plane of Fig. 1.

The transformation

$$t=-\frac{1}{2}\left(z+\frac{1}{z}\right) \tag{1}$$

maps the upper left-hand quadrant of the circle in the z plane onto the upper right-hand quadrant of the t plane of Fig. 1. Here corresponding points on the boundaries of the two regions are given the same alphabetical name. Then the upper right-hand quadrant of the t plane is mapped onto the upper half of the w plane by the transformation

$$w=t^2 \tag{2}$$

Here again, the same letter is used to denote corresponding points. The capacitance C_0 of the coaxial structure is four times the capacitance, in the upper half w plane, between the line segment fa and the infinite line segment, bg. This capacitance, C, is given by the well-known formula [2, p. 58]

$$C=\frac{K(k)}{K'(k)} \tag{3}$$

where, in our case,

$$k^2=\frac{(a-f)(g-b)}{(g-a)(b-f)}. \tag{4}$$

Finally,

$$C_0=4\frac{K(k)}{K'(k)}. \tag{5}$$

In this paper, two series for C_0, one in terms of δ and the other in terms of $\rho=1-\delta$, are given which have certain theoretical and practical advantages. Not only do the series give the limiting behavior of C_0 as δ approaches 0 or 1 but they permit the direct calculation of C_0 with sufficient accuracy for most engineering applications without any resort to elliptic functions.

II. Analysis

If the values for a, b, f, and g, given in the w plane of Fig. 1, are substituted in (4), it is found that

$$k^2=\frac{8\delta^2(1+\delta^4)}{(1+\delta^2)^4}. \tag{6}$$

The values of C_0 given in the middle row of Table I were obtained by finding k^2 from (6), for the given values of δ, and then calculating the complete elliptic integrals, K and K' of (5), using Landen's transformation. To obtain an expansion for C_0 in terms of δ, one may expand (6) in terms of δ^2 to get

$$k^2=8\delta^2(1-4\delta^2+11\delta^4-24\delta^6+45\delta^8-76\delta^{10}+\cdots) \tag{7}$$

and then substitute in

$$\pi\frac{K'}{K}=\ln\frac{16}{k^2}-\frac{1}{2}\left(k^2+\frac{13}{32}k^4+\frac{23}{96}k^6+\frac{2701}{16384}k^8+\cdots\right) \tag{8}$$

so that finally

$$\pi\frac{K'}{K}=\ln 2-2\ln\delta-\frac{1}{8}\left(\delta^8+\frac{13}{32}\delta^{16}+\frac{23}{96}\delta^{24}+\frac{2701}{16384}\delta^{32}+\cdots\right). \tag{9}$$

Determining the coefficent of δ^{32} in (9) in this way requires the coefficents in (8) up to the coefficent of k^{32}. Those not given in [3, p. 1219] are provided in the Appendix.

In order to find the expansion for C_0 in powers of ρ, find k'^2 from (6) and replace δ with $1-\rho$. Then

$$k'^2=\rho^4\frac{1-\rho/2}{1-\rho+\rho^2/2} \tag{10}$$

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